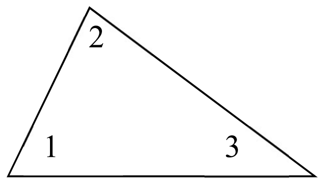
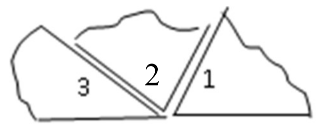
Inductive and Deductive Reasoning

An Example: Using Informal (Inductive) Reasoning and Logical (Deductive) Reasoning

Perhaps you recall the following informal exploration that is sometimes used to show that the angles of a triangle sum to 180 degrees (from Class Activity 10D):

Carefully sketch a triangle on a piece of paper. Use a ruler to make nice straight sides.
Label each corner of the triangle with a number: 1, 2, and 3, as shown below left.



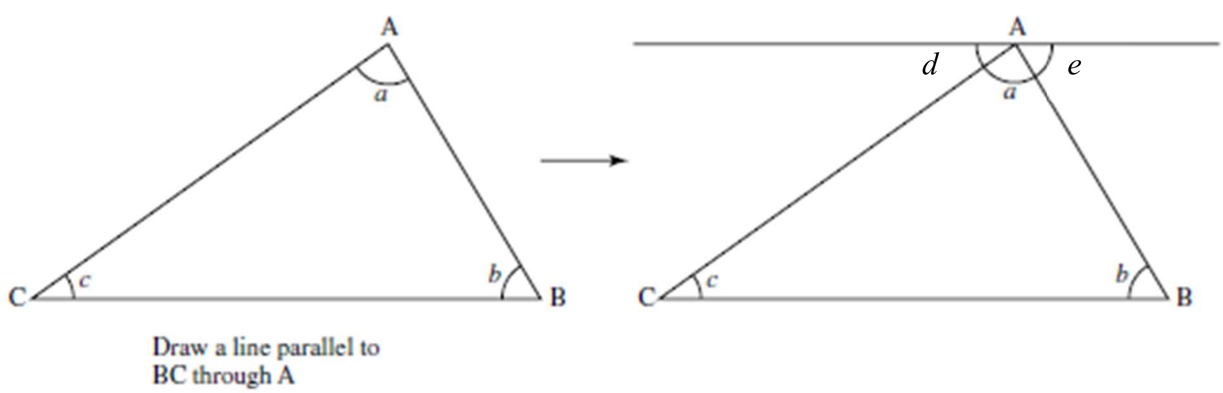


Now carefully tear off the corners, and lay them down so that all three corners meet at one point, as shown above right.

What are the students intended to figure out from this activity?
Is this a valuable experience? Explain?
Is this a proof?

Now compare/contrast with the reasoning expected in the following problem (from Class Activity 10E):

What can you say about the 3 adjacent angles at A that are formed by the triangle and the line through A?
What can you conclude about the sum of the angles in the triangle?



Recognizing Inductive and Deductive Reasoning

When using *Inductive Reasoning* we observe a pattern of events and anticipate that the pattern will continue.

> I expect that pressing the "on" switch will turn on the television.
>
> I always slow down when I see a speed limit sign labeled "photo enforced" because there's probably a speed camera a few feet further.
>
> If I don't remember to call my mother within a few hours of leaving her home I am sure she will call me to make sure I got home safely.

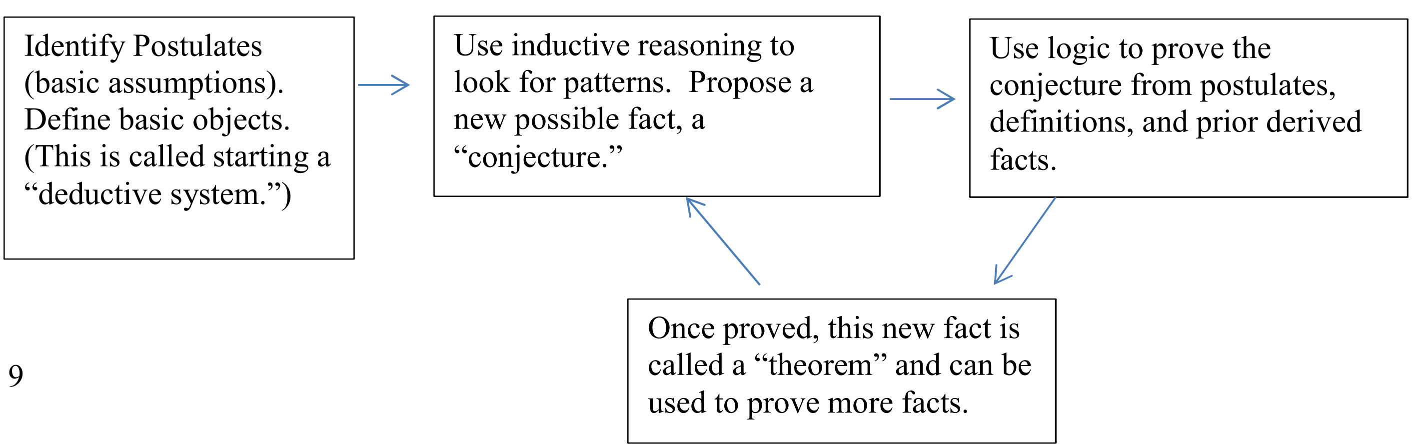
When using *Deductive Reasoning* we start with known facts or principles and use logic to derive new information.

> I know that adding salt to food makes it more flavorful. If I try a new recipe and it tastes bland, I add salt.
>
> I know my niece likes music by Owl City. He just released a new CD. She will like the new CD.
>
> All men are mortal. Socrates is a man. Therefore Socrates is mortal.

Sometimes the two are distinguished this way: Inductive reasoning starts with specific examples and generalizes. Deductive reasoning starts with general principles and applies them to a specific case. Both have challenges. Inductive reasoning sometimes leads to incorrect generalizations based on inadequate evidence. Deductive reasoning requires consistent starting principles and good sound logic or a wrong conclusion will be reached.

In mathematics, both types of reasoning are used. Inductive reasoning is used to examine specific examples and look for patterns that might be true all the time--for example, tearing off the corners of a triangle and noting that the three angles seem to form a straight angle. Deductive reasoning is then used to try to justify, or prove, that the pattern will be consistent. When we claimed that some of the angles were congruent because they were alternate interior angles formed by parallel lines we were using deductive reasoning.

The general starting principles used in math are called "postulates" or "axioms." They must be assumed, or it is not possible to begin proving anything. Applying good logical deduction we can derive new facts, called "theorems." Here's a diagram to help illustrate the process:



Limitations of Inductive Reasoning

Example 1:

Find the following products:

112 x 124	312 x 221	411 x 102
211 x 421	213 x 122	114 x 201

What pattern is suggested?

Now try the following pair:

113 x 223

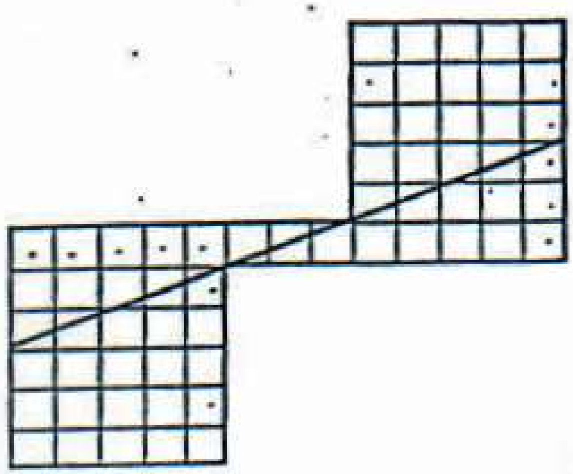
311 x 322

Example 2:

How many square units of area are contained in the design below?

Cut around the outside, and then across the diagonal. Rearrange the four pieces you get into a rectangle. What are the dimensions (length and width) of the rectangle? What is its area?

What is strange about your result? Why is this happening?



Our Foundation

We are now ready to build a "deductive system." This requires that certain facts be assumed in order to allow us to get started. We will take as our starting facts the list of axioms below.

In addition, we will need to make use of the definitions listed on the following pages.

Finally, we can also use the facts we have already proved; these are our first theorems.

Any of these – axioms, definitions, and previously proved theorems -- might be facts we use as "reasons" when writing a statement in a proof.

Our goal is to prove some common properties about isosceles triangles, isosceles trapezoids, and parallelograms using just these starting facts and good logic.

Axioms Summary

These first assumptions will be helpful when we want to add or extend a line in a diagram.

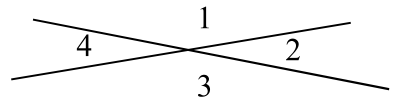
1. Two points determine a line (i.e., you can only draw one line through two points).
2. A line can be extended indefinitely.
3. Given the line *l* and a point P not on *l*, there is exactly one line through P parallel to *l*.
4. In a plane, exactly one perpendicular can be drawn from a point to a line.
5. In a plane, exactly one perpendicular line can be drawn to a point on a line.
6. An angle has exactly one bisector.
7. A segment has exactly one midpoint.

These additional assumptions will help us complete proofs:

8. All right angles are congruent.
9. "Substitution" or "Transitive Property": Things which are equal to the same thing are also equal to each other.
10. Algebra: We can use the rules of algebra to solve equations (e.g. we can subtract the same amount from both sides of an equation).
11. "Corresponding Angles Property": Line *l* is parallel to line *m* if and only if corresponding angles are congruent.*
12. "Reflexive Property": A segment is congruent to itself; an angle is congruent to itself.
13. SSS: If the corresponding sides of two triangles are congruent then the triangles are congruent.
14. ASA: If two pairs of corresponding angles and the included sides are congruent then the triangles are congruent.
15. SAS: If two pairs of corresponding sides and the included angles are congruent then the triangles are congruent.

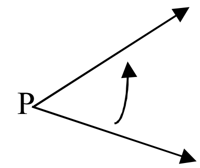
Note the phrase "if and only if" in this property. Remember this means we are actually assuming **two facts. If might be helpful to write them out as separate statements for reference.*

Definitions



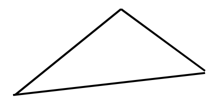
straight angle: An angle measuring 180°. It forms a straight line.
(ex: $\angle 1 + \angle 4$)

vertical angles: When two lines intersect, the angles opposite each other are called vertical angles. (ex: $\angle 1$ and $\angle 3$)



angle of rotation*:* an amount of rotation about a fixed point. One full rotation has 360°. Angles surrounding a point sum to 360° because they constitute one full rotation.

angle formed by two rays with common endpoint P*:* the smallest amount of counterclockwise rotation about P needed to rotate one of the rays to the position of the other ray.



triangle: A closed shape in a plane consisting of three line segments.

congruent angles*:* two angles that have the same measure

line segment

ray

right angle: equal to 90°

acute angle

obtuse angle

*complementary angles—add to 90°

*supplementary angles—add to 180°

Perpendicular: two lines that meet to form right angles

Parallel: two lines in a plane that never meet

*congruent segments—have the same length

*midpoint—the point on a line segment that divides it into two congruent segments

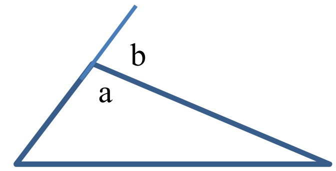
*segment bisector—a point or line that divides a segment into two congruent segments

*angle bisector—a ray that divides an angle in half, i.e. into two congruent angles

right triangle

equilateral triangle: has three equal sides

isosceles triangle: has at least two equal sides quadrilateral



polygon

vertex: corner point where two sides meet

interior angle: angle at a vertex, inside the polygon (a)

exterior angle: angle at a vertex formed by extending one side and measuring to the adjacent side. (b)

square: quadrilateral with four equal sides and four right angles

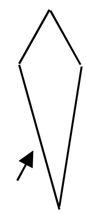
rectangle: quadrilateral with four right angles

rhombus: quadrilateral with four equal sides

parallelogram: quadrilateral with both pairs of opposite sides parallel

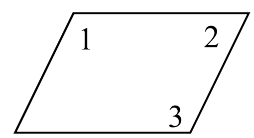
trapezoid: quadrilateral with at least one pair of opposite sides parallel

*kite—a quadrilateral with two non-overlapping pairs of congruent adjacent sides (ex)



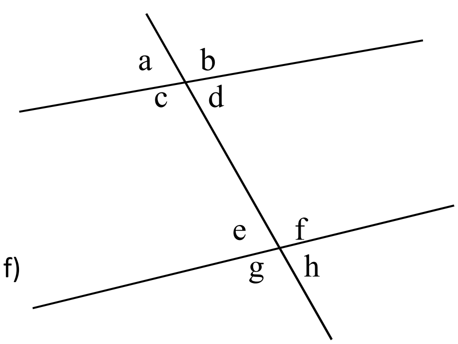
Diagonal: line segment connecting two non-adjacent vertices (corners) of a polygon

*opposite angles (in a quadrilateral)—angles that do not share a side (ex: $\angle 1$ and $\angle 3$)



*consecutive angles (in a polygon)—angles that share a side (ex: $\angle 1$ and $\angle 2$)

Angles formed by (possibly parallel) lines crossed by a transversal:



Corresponding angles (a and e, b and f, c and g, d and h)

Alternate interior angles (c and f, d and e)

Alternate exterior angles (a and h, b and g)

Same-side interior angles (aka consecutive angles) (c and e, d and f)

Circle: the collection of all the points in a plane that are a certain fixed distance (the "radius") away from a certain fixed point (the "center") in the plane.

Theorems (we will add to this list!)

1. Vertical angles (opposite angles) are congruent.

2. If lines are parallel, then alternate interior angles are congruent.

3. If lines are parallel, then same-side interior angles add to 180° (i.e., “are supplementary”).

4. The sum of the angles in a triangle is 180°.

5. The sum of the interior angles in an n-gon is $180(n - 2)$ or $180n - 360$.

6. The sum of the exterior angles of a convex polygon is 360°

7.

8.

9.

A Short Digression . . .

The axioms we have listed guarantee we will be proving theorems in a particular type of geometry called Euclidean geometry (first described by a Greek mathematician named Euclid). For centuries mathematicians thought this was the only possible type of geometry.

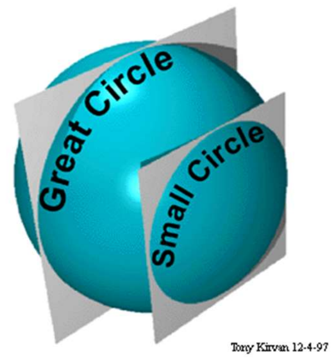
They were always troubled by one particular postulate, however:

Given the line l and a point P not on l, there is exactly one line through P parallel to l.

This seemed too complex to be a postulate, but efforts to prove it as a theorem consistently failed.

In the 19th century several mathematicians made a breakthrough discovery: this axiom is independent of the others. In other words, it is possible to choose a contradictory version (e.g., "There are NO parallel lines through P") and create a different type of geometry, a non-Euclidean geometry.

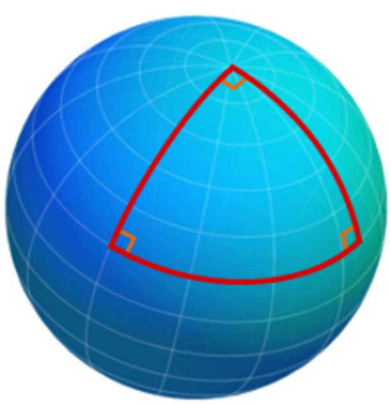
In fact, one very practical non-Euclidean geometry is the geometry of the surface of a sphere.



On a sphere, the shortest distance between two points is along a great circle, a circle that cuts the globe in half – the biggest possible circle on the sphere. This is because, being larger, the circle has a more gradual curvature (it doesn't "bump out" as much. You may have heard of airplanes following the "great circle route" for this reason.)

Being the shortest distance between two points, great circles are the "lines" in spherical geometry.

But since they always cut the globe in half, it is impossible to have two non-intersecting great circles. In other words, there are NO parallel lines in spherical geometry.



This has some interesting implications.

For example, imagine a triangle on the surface of a sphere, such as the one shown at left.

What is the angle sum of this triangle?

It turns out the angle sum of every triangle in spherical geometry is greater than 180 degrees.

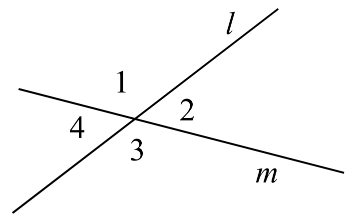
(Recall that when we proved that triangles in Euclidean geometry have exactly 180 degrees we relied on drawing a parallel line and claiming alternate interior angles are congruent. Since there are no parallel lines in spherical geometry, that argument doesn't work.)

Einstein used a different type of non-Euclidean geometry when developing his general theory of relativity. It is now believed our universe may actually be non-Euclidean. However, on the small scale of our everyday experience Euclidean geometry is a very good model.

Writing Formal Proofs

We have been writing proofs already in this course, but we may not have identified them by that name. For example, very early on we justified the following fact:

Vertical Angles Are Congruent



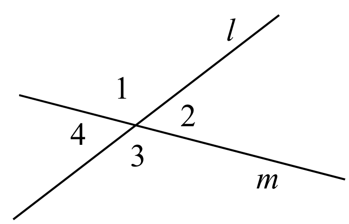
Begin by drawing two intersecting lines, *l* and *m*, as shown at right.
Note that $\angle 1$ and $\angle 3$ are vertical angles.
$\angle 1 + \angle 2 = 180°$ because they form a straight angle.
$\angle 3 + \angle 2 = 180°$ for the same reason.
Since both of these sums equal 180°, they must equal each other: $\angle 1 + \angle 2 = \angle 3 + \angle 2$
Then we can subtract the same amount from both sides of the equation, so subtract $\angle 2$ and you get $\angle 1 = \angle 3$, that is, the angles are congruent.

Sometimes the reasoning used in a proof is arranged in a "two-column" format. You may have used this format in a high school geometry course. Compare and contrast the proof below with the paragraph proof above. What similarities do you notice? Differences? What are the advantages and disadvantages of each format?

Proof: Vertical angles are congruent.

Given: Lines *l* and *m* meet to form angles 1, 2, 3, and 4 as shown.

Prove: $\angle 1 = \angle 3$ (equivalently, $\angle 1 \cong \angle 3$)



Statement	Reason
1. Lines *l* and *m* meet to form angles 1, 2, 3, and 4	1. given
2. $\angle 1 + \angle 2 = 180°$ $\angle 3 + \angle 2 = 180°$	2. def'n straight angle
3. $\angle 1 + \angle 2 = \angle 3 + \angle 2$	3. substitution
4. $\angle 1 = \angle 3$	4. algebra

Note that a proof is nothing more (and nothing less) than the good logical reasoning that convinces us a fact must be true all the time. For the sake of organizing our thoughts, we may use a particular type of set-up, shown above, to summarize our reasoning; don't be overwhelmed by this--it's just a "framework" to organize our observations.

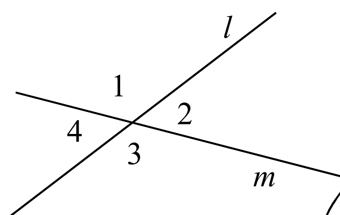
Note also that the informal process we went through earlier in the course is a critical prerequisite to writing a formal proof. Before you write the "organized" version of any proof, whether that will be a paragraph or two columns, take time to convince yourself and the others in your group that your conjecture must always be true.

The following diagram may help you to keep in mind the important elements of a proof. Even if you write your formal proof in paragraph form, these elements need to be present to form a clear convincing argument.

start with a specific diagram and notation, but keep the information as general as possible

Given: Lines *l* and *m* meet to form angles 1, 2, 3, and 4 as shown

To Prove: $\angle 1 = \angle 3$



always start with what you know about this situation (your "starting point")

Statement	**Reason**
1. Lines *l* and *m* meet to form angles 1, 2, 3, and 4 as shown	**1. given**
2.	**2**.
3.	**3**.
___. $\angle 1 = \angle 3$	___.

List the known facts that convinced you this is a true fact. Follow a logical order--you want to convince others! Give reasons for each step, from postulates, definitions, or previously proved theorems.

the last statement will always be what you wanted to prove (your "destination")